15^2 problems in number theory

by

Dominik Burek

Problem 1. Let a, b be a positive integers and p, q distinct primes such that $aq \equiv 1 \pmod{p}$ and $bp \equiv 1 \pmod{q}$. Prove that

$$\frac{a}{p} + \frac{b}{q} > 1.$$

Problem 2. Let a, b be odd positive integers such that $a^b b^a$ is a perfect square. Prove that ab is a perfect square.

Problem 3. Let a, b, c be a positive integers such that $\frac{a\sqrt{2}+b}{b\sqrt{2}+c}$ is rational. Prove that $a+b+c \mid ab+bc+ca$.

Problem 4. Let a be a 8-digit number. Call b the number which was made by moving last digit of a to the beginning. Prove that if $101 \mid a$, then $101 \mid b$.

Problem 5. Prove that there exists infinitely many triples (a, b, c) of positive integers such that

$$a^3 + 3b^6 = c^2.$$

Problem 6. Find all positive integers n for which $n^3 - 7n$ is a square.

Problem 7. Let a, b be a positive integers such that $a + b \mid ab$. Prove that $\gcd(a, b) \geq \sqrt{a+b}$.

Problem 8. Let a, b be a positive integers such that a^2+2b+1 and b^2+2a+1 are squares. Prove that $a = b$.

Problem 9. Let m and n be positive integers. Show that $25m + 3n$ is divisible by 83 if and only if so is $3m + 7n$.

Problem 10. Let p, q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least 3 natural numbers greater than 1 (not necessarily different).

Problem 11. Denote by $d(n)$ the number of all positive divisors of a natural number n (including 1 and n). Prove that there are infinitely many n, such that $\frac{n}{d(n)}$ is an integer.

Problem 12. Let a, b, c, and d be positive integers such that $ab = cd$. Show that there exists positive integers p, q, r, s such that

$$a = pq, \quad b = rs, \quad c = ps, \quad d = qr.$$

Problem 13. Prove that for prime $p > 3$ the following holds

$$p^2 \mid 1 + \frac{1}{2} + \ldots + \frac{1}{p-1}.$$

Problem 14. Let a, b, c, d be positive integers such that $ab = cd$. Prove that $a + b + c + d$ is a composite number.

Problem 15. Prove that $2 \mid \binom{2^n}{k}$ for all $0 < k < 2^n$.

Problem 16. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression of integers such that $i|a_i$ for $i = 1, 2, \ldots, n-1$ and $n \nmid a_n$. Prove that n is a prime power.

Problem 17. Find all pairs of positive integers (a, b) such that $a - b$ is a prime number and ab is a perfect square.

Problem 18. Find all positive integers n such that the decimal representation of n^2 consists of odd digits only.

Problem 19. Let $p \neq 3$ be a prime number. Show that there is a non-constant arithmetic sequence of positive integers $x_1, x_2, \ldots, x_p$ such that the product of the terms of the sequence is a cube.

Problem 20. Are there exist four different positive integers a, b, c, d with $ad = bc$ and $n^2 \leq a, b, c, d < (n+1)^2$ for some positive integer n?

Problem 21. A prime number $p > 2$ and $x, y \in \left\{1, 2, \ldots, \frac{p-1}{2}\right\}$ are given. Prove that if $x\,(p-x)\,y\,(p-y)$ is a perfect square, then $x = y$.

Problem 22. Let b, c be integers and $f(x) = x^2 + bx + c$ be a trinomial. Prove, that if for integers k_1, k_2 and k_3 values of $f(k_1)$, $f(k_2)$ and $f(k_3)$ are divisible by integer $n \neq 0$, then product $(k_1 - k_2)(k_2 - k_3)(k_3 - k_1)$ is divisible by n too.

Problem 23. Given are two integers $a > b > 1$ such that $a + b \mid ab + 1$ and $a - b \mid ab - 1$. Prove that $a < \sqrt{3}b$.

Problem 24. Find all positive integers n which have exactly $\sqrt{n}$ positive divisors.

Problem 25. Prove that for any prime number $p > 3$ exist integers x, y, k that meet conditions: $0 < 2k < p$ and $kp + 3 = x^2 + y^2$.

Problem 26. Suppose that $2^n + 1$ is an odd prime for some positive integer n. Show that n must be a power of 2.

Problem 27. Show that there exist infinitely many positive integers n such that $n^2 + 1$ divides $n!$.

Problem 28. Let n be a positive integer. Show that the product of n consecutive positive integers is divisible by $n!$.

Problem 29. Let m, n, d be a positive integers such that $d \mid m^2n + 1$ and $d \mid mn^2 + 1$. Prove that $d \mid m^3 + 1$ and $d \mid n^3 + 1$.

Problem 30. Find all integers $n \geq 1$ such that $1+2^n+4^n \mid 1+2^{n+1}+4^{n+1}$.

Problem 31. Let k, l, r, s be a positive integers such that $k^r = l^s$. Prove that k and l are powers of some positive integer.

Problem 32. Let $F(k)$ be a product of all positive divisors of a positive integer k. Are there exists positive integers $m \neq n$ such that $F(m) = F(n)$?

Problem 33. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of positive integers (m, n) with $n \geq m \geq 1$.

Problem 34. Show that if $n \geq 6$ is composite, then n divides $(n-1)!$.

Problem 35. Let p be a prime with $p > 5$, and let

$$S = \{p - n^2 | n \in \mathbb{N}, n^2\}.$$

Prove that S contains two elements a and b such that $a|b$ and $1 < a < b$.

Problem 36. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a and b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Problem 37. Show that for every natural number n the product

$$\left(4-\frac{2}{1}\right)\left(4-\frac{2}{2}\right)\left(4-\frac{2}{3}\right)\cdots\left(4-\frac{2}{n}\right)$$

is an integer.

Problem 38. prime number p and integers x, y, z with $0 < x < y < z < p$ are given. Show that if the numbers x^3, y^3, z^3 give the same remainder when divided by p, then $x^2 + y^2 + z^2$ is divisible by $x + y + z$.

Problem 39. Find all the natural numbers a, b, c such that:

- $a^2 + 1$ and $b^2 + 1$ are primes,
- $(a^2 + 1)(b^2 + 1) = (c^2 + 1)$.

Problem 40. Let a, b, c, d be non-zero integers, such that the only quadruple of integers (x, y, z, t) satisfying the equation

$$ax^2 + by^2 + cz^2 + dt^2 = 0$$

is $x = y = z = t = 0$. Does it follow that the numbers a, b, c, d have the same sign?

Problem 41. Let x, y, z be positive integers such that $\frac{x+1}{y} + \frac{y+1}{z} + \frac{z+1}{x}$ is an integer. Let d be the greatest common divisor of x, y and z. Prove that $d \leq \sqrt[3]{xy + yz + zx}$.

Problem 42. Let k be a positive integer. Show that exists positive integer n, such that sets $A = \{1^2, 2^2, 3^3, \ldots\}$ and $B = \{1^2 + n, 2^2 + n, 3^2 + n, \ldots\}$ have exactly k common elements.

Problem 43. Solve the equation $S(2^n) = S(2^{n+1})$.

Problem 44. Compute

$$S(9 \cdot 99 \cdot \ldots \cdot \underbrace{99\ldots9}_{2^n}).$$

Problem 45. Suppose that $a^2 \equiv 1 \pmod{n}$. Then, $n \mid a + b$ if and only if $n \mid ab + 1$.

Problem 46. Let $n \geq 2$ be a positive integer, with divisors

$$1 = d_1 < d_2 < \cdots < d_k = n\ .$$

Prove that

$$d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$$

is always less than n^2.

Problem 47. Find all positive integers n such that $\lfloor \sqrt{n} \rfloor$ divides n.

Problem 48. Prove that the number $512^3 + 675^3 + 720^3$ is composite.

Problem 49. Let the sum of the first n primes be denoted by S_n. Prove that for any positive integer n, there exists a perfect square between S_n and S_{n+1}.

Problem 50. Prove that for any positive integers a and b

$$\left|a\sqrt{2} - b\right| > \frac{1}{2(a+b)}.$$

Problem 51. Show that for all positive integer n,

$$n = \sum_{d|n} \phi(d).$$

Problem 52. Let m, n be a positive integers such that $mn \mid m^2 + n^2 + m$. Show that m is a perfect square.

Problem 53. Let a, b, c, be a positive integer such that $a^2 + b^2 = c^2$. Prove that $\frac{1}{2}(c-a)(c-b)$ is a perfect square.

Problem 54. Determine all positive rational numbers $r \neq 1$ such that $\sqrt[r-1]{r}$ is rational.

Problem 55. Find all positive integers m and n for which

$$1! + 2! + 3! + \cdots + n! = m^2.$$

Problem 56. Let a, b be a positive integers such that $a + b + 1$ is a prime divisor of $4ab - 1$. Prove that $a = b$.

Problem 57. Each integer has one of three colours. Prove that there exists two distinct integers of the same colour with difference being a square of integer.

Problem 58. Prove that for any polynomial f with integer coefficients and integer n we have that $f(n) \mid f(n + f(n))$.

Problem 59. Determine all positive integer x,y,z such that
$$1 = \frac{2}{x^2} + \frac{3}{y^2} + \frac{4}{z^2}.$$

Problem 60. Prove that for any coprime positive integers k, n ($k < n$), the number $\binom{n-1}{k-1}$ is divisible by k.

Problem 61. Let a, b, c, d, e, f be a positive integers such that
$$a + b = c + d = e + f = 101.$$
Prove that $\frac{ace}{bdf}$ can not be expressed as a fraction $\frac{m}{n}$ where m, n are positive integers with sum smaller than 101.

Problem 62. Let a, b, c be a positive integers such that $b \mid a^3$, $c \mid b^3$ and $a \mid c^3$. Prove that $abc \mid (a+b+c)^{13}$.

Problem 63. Let n be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ ($k \geq 2$) be distinct integers in the set $1, 2, \ldots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Problem 64. Determine all positive integers n for which $n^{10} + n^5 + 1$ is a prime.

Problem 65. Let a and b be natural numbers with $a > b$ and having the same parity. Prove that the solutions of the equation
$$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0$$
are natural numbers, none of which is a perfect square.

Problem 66. Positive integers x, y, z are coprime and satisfy equation
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$
Prove that $x + y$ is a perfect square.

Problem 67. Prove that $n \geq 2$ is composite if and only if there exist integers $a, b, x, y \geq 1$ such that

- $a + b = n$,
- $\frac{x}{a} + \frac{y}{b} = 1$.

Problem 68. Let $k \geq 2$ be an integer and consider integers $a_1, a_2, \ldots, a_n$ such that

$$a_1 + 2^i a_2 + 3^i a_3 + \ldots + n^i a_n = 0 \quad \text{for } i = 1, 2, \ldots, k-1.$$

Prove that $a_1 + 2^k a_2 + 3^k a_3 + \ldots + n^k a_n$ is divisible by $k!$.

Problem 69. Let $n \geq 3$ be an integer. Prove that, the sum of cubes of all positive integer such that $\gcd(k, n) = 1$ and $k < n$, is divisible by n.

Problem 70. Let k, $n > 1$ be integer such that $p = 2k - 1$ is a prime. Prove that if the number

$$\binom{n}{2} - \binom{k}{2}$$

is divisible by p then is divisible by p^2 too.

Problem 71. Prove that for any integer $n \geq 2$ and any prime p the number $n^{p^p} + p^p$ is composite.

Problem 72. Determine all polynomials W with integer coefficients, such that for any positive integer n the number $2^n - 1$ is divisible by $W(n)$.

Problem 73. Let $n \geq 2$ be an integer. Let $r_1, r_2, r_3, \ldots, r_{n-1}$ are residues of

$$1,\ 1+2,\ 1+2+3,\ \ldots,\ 1+2+\ldots+(n-1)$$

modulo n. Determine all n, for which $(r_1, r_2, \ldots, r_{n-1})$ is a permutation of $(1, 2, \cdots, n-1)$.

Problem 74. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2 + b + c, b^2 + c + a, c^2 + a + b$ to be perfect squares.

Problem 75. Let n and q be integers with $n \geq 5$, $2 \leq q \leq n$. Prove that $q-1$ divides $\left\lfloor \frac{(n-1)!}{q} \right\rfloor$.

Problem 76. Let n be a positive integer with $n > 1$. Prove that

$$\frac{1}{2} + \ldots + \frac{1}{n}$$

is not an integer.

Problem 77. Prove that any rational number may be written as

$$\frac{a^2 + b^3}{c^5 + d^7},$$

where a, b, c, d are positive integers.

Problem 78. Consider a sequence $a_n = |n(n+1) - 19|$ for $n = 0, 1, 2, \ldots$. Prove that for any $n \neq 4$ the following holds: If for all integers $k < n$ numbers a_k and a_n are coprime, then a_n is a prime.

Problem 79. Prove that there exist infinitely many even positive integers k such that for every prime p the number $p^2 + k$ is composite.

Problem 80. Let a, b be a positive integers such that ab and $(a+1)(b+1)$ are perfect squares. Prove that there exists integer $n > 1$ such that $(a+n)(b+n)$ is a perfect square.

Problem 81. Show that $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.

Problem 82. A prime p and a positive integer n are given. The product

$$(1^3 + 1)(2^3 + 1) \ldots (n^3 + 1)$$

is divisible by p^3. Prove that $p \leq n + 1$.

Problem 83. Prove that there exists infinitely many positive integers a, b such that

$$\gcd(a^2 + 1, b^2 + 1) = a + b.$$

Problem 84. Let a, b, c, d be a positive integers such that a^2, b^2 give residues c, d modulo $a+b+1$, respectively and c^2, d^2 give residues a, b modulo $c+d+1$, respectively. Prove that $|a-d| = |b-c|$.

Problem 85. Let b, n be integers greater than 1 such that for all $k > 1$ one can find an integer a such that $k \mid b - a^n$. Prove that b is n-th power of an integer.

Problem 86. Integers a, b and rational numbers x, y satisfy $y^2 = x^3 + ax + b$. Prove that we can write $x = \frac{u}{v^2}$ and $y = \frac{w}{v^3}$ for some integers u, v, w, with $\gcd(u, v) = \gcd(w, v) = 1$.

Problem 87. Prove that there are infinitely many 5-tuples (a, b, c, d, e) such that $a \mid b^2 - 1$, $b \mid c^2 - 1$, $c \mid d^2 - 1$, $d \mid e^2 - 1$ and $d \mid a^2 - 1$.

Problem 88. Let $p > 3$ is a prime number and $k = \lfloor \frac{2p}{3} \rfloor$. Prove that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by p^2.

Problem 89. Find all positive integers m, n such that

$$\binom{m}{n} = 1984.$$

Problem 90. Prove that every integer can be written as a sum of 5 perfect cubes.

Problem 91. Find all integer solutions of the following equation

$$x^2 + y^2 + z^2 = 2xyz.$$

Problem 92. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

Problem 93. Let c be a fixed natural number. Sequence (a_n) is defined by:

$$a_1 = 1,\ a_{n+1} = d(a_n) + c \text{ for } n = 1, 2, \ldots,$$

where $d(m)$ is number of divisors of m. Prove that there exist k natural such that the sequence $a_k, a_{k+1}, \ldots$ is periodic.

Problem 94. Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

Problem 95. Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

Problem 96. Show that, for any fixed integer $n, k \geq 1$, the sequence

$$k,\ k^k,\ k^{k^k},\ k^{k^{k^k}}, \ldots \pmod{n}$$

is eventually constant.

Problem 97. Find all pairs (a, b) of integers satisfying: there exists an integer $d \geq 2$ such that $a^n + b^n + 1$ is divisible by d for all positive integers n.

Problem 98. Call a positive integer n a *good number*, if there exists prime number p such that $p \mid n$ and $p^2 \nmid n$. Prove that 99% numbers among $1, 2, 3, \ldots, 10^{12}$ are good.

Problem 99. Let k, n be odd positive integers greater than 1. Prove that if there a exists natural number a such that $k \mid 2^a + 1,\ n \mid 2^a - 1$, then there is no natural number b satisfying $k \mid 2^b - 1,\ n \mid 2^b + 1$.

Problem 100. Determine all positive integers n for which

$$\frac{n^2 + 1}{[\sqrt{n}]^2 + 2}$$

is an integer.

Problem 101. Prove that for any positive integer k, there exists an arithmetic sequence $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_k}{b_k}$ of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$ such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

Problem 102. The ratio of prime numbers p and q does not exceed 2 ($p \neq q$). Prove that there are two consecutive positive integers such that the largest prime divisor of one of them is p and that of the other is q.

Problem 103. Prove that there exist infinitely many positive integer triples (a, b, c) such that

$$a \mid bc - 1, \quad b \mid ac + 1, \quad c \mid ab + 1.$$

Problem 104. Let $S(n)$ denote the sum of the digits of the base-10 representation of an natural number n. Prove that for all primes p, there exists infinitely many n which satisfy $S(n) \equiv n \pmod{n}$.

Problem 105. Let p be a prime and n be a positive integer such that p^2 divides $\prod_{k=1}^{n}(k^2+1)$. Show that $p < 2n$.

Problem 106. Given an integer $m \geq 2$ and m positive integers $a_1, a_2, \ldots, a_m$ prove that there exist infinitely many positive integers n, such that

$$a_1 1^n + a_2 2^n + \ldots + a_m m^n$$

is composite.

Problem 107. Let

$$S = \{n \mid n-1,\ n \text{ and } n+1 \text{ can be expressed as the sum of two squares}\}.$$

Prove that if $n \in S$ then $n^2 \in S$.

Problem 108. The sequence $\{a_n\}_{n\geq 0}$ is defined by $a_0 = 2, a_1 = 4$ and

$$a_{n+1} = \frac{a_n a_{n-1}}{2} + a_n + a_{n-1}$$

for all positive integers n. Determine all prime numbers p for which there exists a positive integer m such that p divides the number $a_m - 1$.

Problem 109. For any positive integer n let $d(n)$ denote the number of positive divisors of n. Do there exist positive integers a and b, such that $d(a) = d(b)$ and $d(a^2) = d(b^2)$, but $d(a^3) \neq d(b^3)$?

Problem 110. Let A and B be disjoint non-empty sets with $A \cup B = \{1, 2, 3, \ldots, 10\}$. Show that there exist elements $a \in A$ and $b \in B$ such that the number $a^3 + ab^2 + b^3$ is divisible by 11.

Problem 111. Let $n \geq 50$ be a natural number. Prove that n is expressible as sum of two natural numbers $n = x + y$, so that for every prime number p such that $p \mid x$ or $p \mid y$ we have $\sqrt{n} \geq p$.

Problem 112. Let a, b be two positive integers and $a > b$.We know that $\gcd(a-b, ab+1) = 1$ and $\gcd(a+b, ab-1) = 1$. Prove that $(a-b)^2+(ab+1)^2$ is not a perfect square.

Problem 113. Let n be a positive integer and p be a prime number such that $np + 1$ is a perfect square. Prove that $n + 1$ can be written as the sum of p perfect squares.

Problem 114. Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa$$

then $a = b = c$.

Problem 115. Determine whether $712! + 1$ is a prime number.

Problem 116. Let $d(k)$ denote the number of positive divisors of a positive integer k. Prove that there exist infinitely many positive integers M that cannot be written as

$$M = \left(\frac{2\sqrt{n}}{d(n)}\right)^2$$

for any positive integer n.

Problem 117. Let a and b be rational numbers such that $s = a+b = a^2+b^2$. Prove that s can be written as a fraction where the denominator is relatively prime to 6.

Problem 118. Let a, b, c, d, e, f be positive integers and let $S = a+b+c+d+e+f$. Suppose that the number S divides $abc+def$ and $ab+bc+ca-de-ef-df$. Prove that S is composite.

Problem 119. Let k, n be a positive integers such that $k > n!$. Prove that there exist distinct prime numbers $p_1, p_2, \ldots, p_n$ such that $p_i \mid k+i$ for all $i = 1, 2, \ldots, n$.

Problem 120. Let k be a positive integer. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = k+1$, $a_{n+1} = a_n^2 - ka_n + k$. Show that a_m and a_n are coprime (for $m \neq n$).

Problem 121. For positive integers $a > b > 1$, define

$$x_n = \frac{a^n - 1}{b^n - 1}$$

Find the least d such that for any a, b, the sequence x_n does not contain d consecutive prime numbers.

Problem 122. Two natural numbers d and d', where $d' > d$, are both divisors of n. Prove that $d' > d + \frac{d^2}{n}$.

Problem 123. Let $n \geq 3$ and consider pairwise coprime numbers $p_1, p_2, \ldots, p_n$. Suppose that for any $k \in \{1, 2, \ldots, n\}$ the residue of $\prod_{i \neq k} p_i$ modulo p_k equals r. Prove, that $r \leq n-2$.

Problem 124. From the interval $(2^{2n}, 2^{3n})$ are selected $2^{2n-1}+1$ odd numbers. Prove that there are two among the selected numbers, none of which divides the square of the other.

Problem 125. A positive integer d is called *nice* iff for all positive integers x, y hold: d divides $(x+y)^5 - x^5 - y^5$ iff d divides $(x+y)^7 - x^7 - y^7$. Prove that exist infinitely many nice numbers.

Problem 126. Determine the smallest natural number $a \geq 2$ for which there exists a prime number p and a natural number $b \geq 2$ such that

$$\frac{a^p - a}{p} = b^2.$$

Problem 127. Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define

$$S(a) = \sum_{i=1}^{n} c_i a_i.$$

Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.

Problem 128. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 129. An integer $n \geq 1$ is called *balanced* if it has an even number of distinct prime divisors. Prove that there exist infinitely many positive integers n such that there are exactly two balanced numbers among $n, n + 1, n + 2$ and $n + 3$.

Problem 130. Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

Problem 131. Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right) \cdots \left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}.$$

Problem 132. Let $n \geq 3$ be an integer such that $4n + 1$ is a prime number. Prove that $4n + 1$ divides $n^{2n} - 1$.

Problem 133. Let a, b be a distinct positive integers such that $(a^2 + ab + b^2)$ divides $ab(a + b)$. Prove that $|a - b| > \sqrt[3]{ab}$.

Problem 134. Let n, m be integers greater than 1, and let $a_1, a_2, \ldots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \ldots, b_m$ not greater than n, such that

$$\gcd(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) < n.$$

Problem 135. Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Problem 136. Let a and b be two positive integers such that $2ab$ divides $a^2 + b^2 - a$. Prove that a is perfect square

Problem 137. Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Problem 138. Determine the maximal possible length of the sequence of consecutive integers which are expressible in the form $x^3 + 2y^2$, with x, y being integers.

Problem 139. Prove that if $x_1, x_2, \ldots, x_{2019}, y_1, y_2, \ldots, y_{2019}$ are positive integers, then the product product

$$(2x_1^2 + 3y_1^2)(2x_2^2 + 3y_2^2) \ldots (2x_{2019}^2 + 3y_{2019}^2)$$

is not a perfect square.

Problem 140. Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Problem 141. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1_1 < \ldots_k$. Prove that e_1 is an even number.
A number n is perfect if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

Problem 142. Is there exist integer greater then 2018^{2018}, which is not of the form $x^2 + y^3 + z^6$ for some positive integers x, y, z?

Problem 143. Let x, y, z, t be a positive integers such that

$$x^2 + y^2 + z^2 + t^2 = 2018!.$$

Prove that $x, y, z, t > 10^{250}$.

Problem 144. Let $p_1, p_2, \ldots, p_6$ be a primes such that $p_{k+1} = 2p_k + 1$ for $k = 1, 2, \ldots, 5$. Prove that

$$15 \mid \sum_{1 \le i < j \le 6} p_i p_j.$$

Problem 145. Prove that for every positive integer n, there exist n positive integers which are cubes and the sum of them is a perfect square and the product of them is a perfect cube.

Problem 146. Prove that

$$\binom{2n}{n} \mid \operatorname{lcm}(1, 2, \ldots, 2n)$$

for any positive integer n.

Problem 147. Determine weather there exists an infinite sequence of positive integers in which none of a term and none of a sum of arbitrary number of terms i a perfect power.

Problem 148. Find all positive integers n for which there exists pairwise distinct positive integers $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n \quad \text{oraz} \quad a_1 \cdot a_2 \cdot \ldots \cdot a_n = b_1 \cdot b_2 \cdot \ldots \cdot b_n.$$

Problem 149. Let a, b, c, d be a positive integers such that $cn + d \mid an + b$ for any positive integers n. Prove that $a = kc$ and $b = kd$ for some integer k.

Problem 150. Let a, b, c be a non-zero integers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

is integer. Prove that abc is a perfect cube.

Problem 151. Suppose that a, b, c are positive integers such that a^b divides b^c, and a^c divides c^b. Prove that a^2 divides bc.

Problem 152. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

Problem 153. Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

Problem 154. Prove that if the numbers m and n are coprime, then the number $(m+n-1)!$ is divisible by $m!n!$.

Problem 155. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1}+y$ and $x+y^{p-1}$ are both powers of p.

Problem 156. Let a, b, c be positive integers such that

$$\frac{ab}{a+b}, \quad \frac{bc}{b+c}, \quad \frac{ca}{c+a}$$

are integers and $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = d$. Prove that

$$d \geq \sqrt{2 \min\{a, b, c\}}.$$

Problem 157. Prime number $p > 3$ is congruent to 2 modulo 3. Let $a_k = k^2+k+1$ for $k = 1, 2, \ldots, p-1$. Prove that product $a_1 a_2 \ldots a_{p-1}$ is congruent to 3 modulo p.

Problem 158. Prove that there are infinitely many positive integers n such that $2^{2^n+1} + 1$ is divisible by n but $2^n + 1$ is not.

Problem 159. Determine if there are 2018 different positive integers such that the sum of their squares is a perfect cube and the sum of their cubes is a perfect square.

Problem 160.

(a) Prove that for every positive integer m there exists an integer $n \geq m$ such that

$$\left\lfloor \frac{n}{1} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdots \left\lfloor \frac{n}{m} \right\rfloor = \binom{n}{m}$$

(b) Denote by $p(m)$ the smallest integer $n \geq m$ such that the equation $(*)$ holds. Prove that $p(2018) = p(2019)$.

Problem 161. For a nonnegative integer n, define a_n to be the positive integer with decimal representation

$$1\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}1.$$

Prove that $\frac{a_n}{3}$ is always the sum of two positive perfect cubes but never the sum of two perfect squares.

Problem 162. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).

Problem 163. Is there a power of 2 such that it is possible to rearrange the digits giving another power of 2?

Problem 164. Prove that there exists infinitely many integers $n > 1$ for which

$$(x+1)^{n+1} - (x-1)^{n+1} = y^n$$

has no integer solutions x, y.

Problem 165. Let p be a prime. Suppose that p is a quadratic mean of some distinct positive integers u and v. Prove that $2p - u - v$ is a square or double square.

Problem 166. Let $1 = d_0 < d_1 < \ldots < d_m = 4k$ be all positive divisors of $4k$, where k is a positive integer. Prove that there exists $1 \leq i \leq m$ such that $d_i - d_{i-1} = 2$.

Problem 167. If $\varphi(n) \mid (n-1)$, then prove that there is no prime p such that $p^2 \mid n$.

Problem 168. Consider all quadruples (p, a, b, c) of positive integers such that p is an odd prime, a, b, and c are distinct, and $ab+1$, $bc+1$ and $ca+1$ are divisible by p. Prove that

$$p+2 \leq \frac{a+b+c}{3}.$$

When does equality occur?

Problem 169. Prove that

$$2^{n+1} \mid \left\lceil (1+\sqrt{3})^{2n} \right\rceil.$$

Problem 170. Find all positive integers a and b for which there are three consecutive integers at which the polynomial

$$P(n) = \frac{n^5+a}{b}$$

takes integer values.

Problem 171. Prove that there exist infinitely many positive integers n such that the largest prime divisor of n^4+n^2+1 is equal to the largest prime divisor of $(n+1)^4+(n+1)^2+1$.

Problem 172. For a composite number n, let d_n denote its largest proper divisor. Show that there are infinitely many n for which $d_n + d_{n+1}$ is a perfect square.

Problem 173. Define a function $f(n)$ from the positive integers to the positive integers such that $f(f(n))$ is the number of positive integer divisors of n. Prove that if p is a prime, then $f(p)$ is prime.

Problem 174. Prove that for any positive integers $a_1 < a_2 < \ldots < a_n$ the following inequality holds

$$\frac{1}{a_1} + \frac{1}{\operatorname{lcm}(a_1, a_2)} + \frac{1}{\operatorname{lcm}(a_1, a_2, a_3)} + \ldots + \frac{1}{\operatorname{lcm}(a_1, a_2, \ldots, a_n)} < 2.$$

Problem 175. Let n be a positive integer and take prime p such that $n < p < \frac{4}{3}n$. Prove that

$$p \mid \sum_{k=0}^{n} \binom{n}{k}^4.$$

Problem 176. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example, $\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$.

Problem 177. Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Problem 178. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

Problem 179. Let S be the set of all positive integers that are not perfect squares. For n in S, consider choices of integers $a_1, a_2, \dots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Problem 180. Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, 4, \cdots \right\}.$$

(a) Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

(b) For every integer $x \geq 2$ let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different. Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2$, $y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$).

Problem 181. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\left\{1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right\}.$$

Problem 182. Find all integers x, y, z, t such that

$$x^2 + 6y^2 = z^2, \qquad 6x^2 + y^2 = t^2.$$

Problem 183. Prove that any positive integer n may be represent as

$$n = \pm 1^2 \pm 2^2 \pm \ldots \pm k^2,$$

for some positive integers k and some choice of signs.

Problem 184. Let $k > 1$ be an integer. A sequence $(a_n)_{n=1}^{\infty}$ satisfies for any $n \geq 1$ the following condition

$$\sum_{d|n} d a_d = k^n.$$

Prove that for any $n \geq 1$ the number a_n is integer.

Problem 185. Let p be any prime. Find all pairs (x, y) of integers such that

$$x^3 + y^3 - 3xy = p - 1.$$

Problem 186. Suppose p is a prime and $a_1, a_2, \ldots, a_p$ are primes forming an arithmetic progression with common difference d. If $a_1 > p$, prove that $p \mid d$.

Problem 187. Let n be a positive integers and let d be a positive divisor of $2n^2$. Prove that $n^2 + d$ is not a perfect square.

Problem 188. The numbers p and q are prime and satisfy

$$\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}$$

for some positive integer n. Find all possible values of $q - p$.

Problem 189. Show that there exist infinitely many pairs of positive integers (m, n) for which $(m!)^n + (n!)^m + 1$ is divisible by $m + n$

Problem 190. Let m, n be distinct positive integers. Prove that

$$gcd(m,n) + gcd(m+1, n+1) + gcd(m+2, n+2) \le 2|m-n| + 1.$$

Further, determine when equality holds.

Problem 191. The number $\overline{13\ldots3}$, with $k > 1$ digits 3, is a prime. Prove that $6 \mid k^2 - 2k + 3$.

Problem 192. Prove that if a, b are co-prime integers, and $p \neq 3$ is a prime divisor of $a^3 - 3ab^2 + b^3$ then $p \equiv \pm 1 \pmod 9$.

Problem 193. Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \ge 2b + 2$.

Problem 194. Prove that for every non-negative integer n there exist integers x, y, z with $gcd(x, y, z) = 1$, such that $x^2 + y^2 + z^2 = 3^{2^n}$.

Problem 195. Let a be any integer. Prove that there are infinitely many primes p such that

$$p \,|\, n^2 + 3 \qquad \text{and} \qquad p \,|\, m^3 - a$$

for some integers n and m.

Problem 196. Let $a_1, a_2, \ldots, a_n$ be positive integers with product P, where n is an odd positive integer. Prove that

$$\gcd(a_1^n + P, a_2^n + P, \ldots, a_n^n + P) \le 2 \gcd(a_1, \ldots, a_n)^n.$$

Problem 197. Let S be the set of all ordered pairs (a, b) of integers with $0 < 2a < 2b < 2017$ such that $a^2 + b^2$ is a multiple of 2017. Prove that

$$\sum_{(a,b)\in S} a = \frac{1}{2} \sum_{(a,b)\in S} b.$$

Problem 198. Let $a_1, \ldots, a_k$ and $m_1, \ldots, m_k$ be integers with $m_1 \geq 2$ and $2m_i \leq m_{i+1}$ for $1 \leq i \leq k-1$. Show that there are infinitely many integers x which do not satisfy the following system of congruences

$$\begin{cases} x \equiv a_1 \pmod{m_1}, \\ x \equiv a_2 \pmod{m_2}, \\ \ldots \\ x \equiv a_k \pmod{m_k}. \end{cases}$$

Problem 199. If an integer n is such that $7n$ is the form $a^2 + 3b^2$, prove that n is also of that form.

Problem 200. Let a, b, n be a positive integers with $a > b$ and $ab - 1 = n^2$. Prove that $a - b \geq \sqrt{4n-3}$ and study the cases where the equality holds.

Problem 201. Let A be the set of the 16 first positive integers. Find the least positive integer k satisfying the condition: In every k-subset of A, there exist two distinct $a, b \in A$ such that $a^2 + b^2$ is prime.

Problem 202. Prove that the following system of equation

$$\begin{cases} a^2 - b = c^2 \\ b^2 - a = d^2 \end{cases}$$

has no positive distinct integer solutions a, b, c, d.

Problem 203. Prove that the equation $x^x = y^3 + z^3$ has infinitely many positive integer solutions x, y, z.

Problem 204. The *palindrome* is a number which reads the same backward as forward. Let (x_n) be an increasing sequence of all palindromes. Find all prime numbers which can divide differences $x_{n+1} - x_n$.

Problem 205. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be a positive integers such that

$$x_6 = 144 \text{ and } x_{n+3} = x_{n+2}(x_{n+1} + x_n) \text{ for } n = 1, 2, 3, 4.$$

Find x_7.

Problem 206. Solve in positive integers the following equation

$$n(n+1)(n+2)(n+3) = m(m+1)^2(m+2)^3(m+3)^4$$

Problem 207. Let a, b and c, be a positive integers such that $\gcd(a, b, c) = 1$ and

$$a^2 + b^2 + c^2 = 2(ab + bc + ca).$$

Prove that all of a, b, c are perfect squares.

Problem 208. Let x, y be a positive integers, such that $x^2 - 4y + 1$ is a multiple of $(x - 2y)(1 - 2y)$. Prove that $|x - 2y|$ is a square number.

Problem 209. Let n be a positive integer. Prove that $2n - 1$ is a prime iff any set of n distinct positive integers contains two a, b such that

$$\frac{a+b}{\gcd(a,b)} \geq 2n - 1.$$

Problem 210. Let a, b, c be positive integers. Prove that

$$\operatorname{lcm}(a, b) \neq \operatorname{lcm}(a + c, b + c).$$

Problem 211. Let $a_1, a_2, \ldots, a_n$ $(n \geqslant 3)$ be a positive integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Moreover, if $s = a_1 + a_2 + \ldots + a_n$, then any number $a_1, a_2, \ldots, a_n$ divides s. Prove that s^{n-2} is divisible by $a_1 a_2 \ldots a_n$.

Problem 212. Let $f(t) = t^3 + t$. Determine, if there are positive rational numbers x, y and positive integers m, n, such that $xy = 3$ and

$$\underbrace{f(f(\ldots f(f(x))\ldots))}_{m \text{ razy}} = \underbrace{f(f(\ldots f(f(y))\ldots))}_{n \text{ razy}}.$$

Problem 213. Find all non-negative integers x, y such that

$$\sqrt{xy} = \sqrt{x+y} + \sqrt{x} + \sqrt{y}.$$

Problem 214. Let n be a positive integer. Prove that there exists positive integers a and b, such that

$$a^2 + a + 1 = (n^2 + n + 1)(b^2 + b + 1).$$

Problem 215. Let a, b, c be positive integers. Prove that there is a positive integer n such that

$$(a^2 + n)(b^2 + n)(c^2 + n)$$

is a perfect square.

Problem 216. Let a, b, c, d be a positive integers such that $ad = b^2 + bc + c^2$. Prove that $a^2 + b^2 + c^2 + d^2$ is composite.

Problem 217. Positive rational number a and b satisfy the equality

$$a^3 + 4a^2b = 4a^2 + b^4.$$

Prove that the number $\sqrt{a} - 1$ is a square of a rational number.

Problem 218. Let n be a positive integer which is a sum of three squares of integers. Prove that n^2 is also a sum of three squares of integers.

Problem 219. Prove that $\dfrac{5^{125} - 1}{5^{25} - 1}$ is a composite number.

Problem 220. Prove that for any positive integer k, the number $\dfrac{7^{7^{k+1}} + 1}{7^{7^k} + 1}$ is composite.

Problem 221. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Problem 222. Let a and n be a positive integers such that n divides $a^2 + 1$. Prove that there exists positive integer b such that $n(n^2 + 1)$ divides $b^2 + 1$.

Problem 223. Let a be a positive integer such that $4(a^n + 1)$ is cube of some integral for any natural number n. Show that $a = 1$.

Problem 224. Let $a \geq 3$ be an integer. Prove that there exists infinitely many positive integers n such that $n^2 \mid a^n - 1$.

Problem 225. Integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \ldots a_n$, then prove that

$$(a_1 a_2 \cdots a_n)^{m-1} \geq (n!)^m.$$